

# On some problems of Euclidean Ramsey theory \*

Shkredov I.D.

Annotation.

*In the paper we prove, in particular, that for any measurable coloring of the euclidian plane into two colours there is a monochromatic triangle with some restrictions on the sides. Also we consider similar problems in finite fields settings.*

## 1 Introduction

Let  $\Pi = \mathbf{R}^2$  be the ordinary euclidian plane. Any partition of  $\Pi$  onto  $k$  disjoint sets  $C_1, \dots, C_k$  is called  $k$ -coloring of  $\Pi$  and the sets  $C_1, \dots, C_k$  are called *colors*. A well-known unsolved question of Euclidian Ramsey Theory (see [3], [10]) asks us about the existence of a monochromatic (that is belonging to the same color) non-equilateral triangle (that is just any three points from  $\Pi$ ) in any two-coloring of the plane. The problem seems to be difficult and only partial results are known, see [10]. In particular, the question remains open even for the case of a degenerate triangle, having all three points lying on a line. A parallel and even more famous problem in the area is to find the *chromatic number* of the plane  $\chi(\mathbf{R}^2)$ , that is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart. It is well-known that  $4 \leq \chi(\mathbf{R}^2) \leq 7$ . In his beautiful paper [4] Falconer proved that if all colors are *measurable* sets then the correspondent *measurable chromatic number* of the plane is at least five. Our paper is devoted to a measurable analog of the considered two-coloring problem. The main result is the following, see Theorem 6 and Theorem 9 from section 3.

**Theorem 1** *Let  $ABC$  be a nondegenerate triangle such that  $|AB|/|AC| = \omega$ . Suppose that*

$$\min_{t \geq 0} (J_0(t) + J_0(\omega t)) \geq -0.5972406,$$

*where  $J_0$  is the zeroth Bessel function. Then any measurable coloring of  $\mathbf{R}^2$  into two colors contains a monochromatic triangle.*

*Further, if*

$$\min_{t \geq 0} (J_0(t) + J_0(\kappa t) + J_0((1 + \kappa)t)) > -1.$$

*Then for any measurable coloring of the plane  $\Pi$  into two colors there is a monochromatic collinear triple  $\{x, y, z\}$  such that  $y \in [x, z]$  and  $\|z - y\|/\|y - x\| = \kappa$ .*

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\*This work was supported by grant Russian Scientific Foundation RSF 14-11-00433.

The proof uses simple Fourier analysis in spirit of paper [9] and hugely relies on the fact that we have deal just with two colors. Also we consider a model situation of the plane over the prime finite field  $\mathbb{F}_p \times \mathbb{F}_p$  and prove (a slightly stronger) analog of Theorem 1, see section 2. The proof develops the method from [1], [6], [12].

The author is grateful to R. Prasolov and J. Wolf for useful discussions.

## 2 Finite fields case

Let  $p$  be a prime number, and  $\mathbb{F}_p$  be the prime field. Let also  $\Pi = \mathbb{F}_p \times \mathbb{F}_p$  be the prime plane. If  $x \in \Pi$  then we write  $x = (x_1, x_2)$ . For any  $j \neq 0$  define a *sphere* in  $\Pi$ , that is the set

$$\mathcal{S}_j = \{x \in \Pi : \|x\| := x_1^2 + x_2^2 = j\}.$$

For any function  $f : \Pi \rightarrow \mathbb{C}$  denote its Fourier transform as

$$\widehat{f}(r) := \sum_{x \in \Pi} f(x) e^{-\frac{2\pi i(x_1 r_1 + x_2 r_2)}{p}} = \sum_{x \in \Pi} f(x) e^{-\frac{2\pi i \langle x, r \rangle}{p}} = \sum_{x \in \Pi} f(x) e(-\langle x, r \rangle).$$

The inverse formula takes place

$$f(x) = p^{-2} \sum_{r \in \Pi} \widehat{f}(r) e(\langle r, x \rangle). \quad (1)$$

For any two functions  $f, g : \Pi \rightarrow \mathbb{C}$  the Parseval identity holds

$$\sum_{x \in \Pi} f(x) \overline{g(x)} = p^{-2} \sum_{r \in \Pi} \widehat{f}(r) \overline{\widehat{g}(r)}. \quad (2)$$

Further, put

$$(f * g)(x) := \sum_{y \in \Pi} f(y) g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \Pi} f(y) g(y + x).$$

Then

$$\widehat{f * g} = \widehat{f} \widehat{g} \quad \text{and} \quad \widehat{f \circ g} = \widehat{f}^c \widehat{g} = \overline{\widehat{f}} \widehat{g}, \quad (3)$$

where for a function  $f : \Pi \rightarrow \mathbb{C}$  we put  $f^c(x) := f(-x)$ . Clearly,  $(f * g)(x) = (g * f)(x)$  and  $(f \circ g)(x) = (g \circ f)(-x)$ ,  $x \in \Pi$ . If  $A \subseteq \Pi$  is a set then denote by  $A(x)$  its characteristic function.

Using Gauss and Kloosterman sums one can prove the following rather standard lemma, see e.g. [6]. Exact formula for the cardinalities of the spheres in  $\Pi$  can be obtained as well.

**Lemma 2** *We have*

$$|\mathcal{S}_j| = p + 2\theta\sqrt{p}, \quad (4)$$

where  $|\theta| \leq 1$ , and for all  $r \neq 0$  one has

$$|\widehat{\mathcal{S}}_j(r)| \leq 2\sqrt{p}. \quad (5)$$

Moreover, for any invertible  $\mathbf{g} : \Pi \rightarrow \Pi$  and all  $r \neq 0$  the following holds

$$|\widehat{\mathbf{g}(\mathcal{S}_j)}(r)| \leq 2\sqrt{p}. \quad (6)$$

**Proof.** We will prove just (6), the proof of (4), (5) is similar and is contained in [6], Lemma 2. Put  $\mathcal{S} = \mathcal{S}_j$ . We have

$$\begin{aligned} \widehat{\mathbf{g}(\mathcal{S})}(r) &= \sum_x \mathcal{S}(\mathbf{g}^{-1}x) e(-\langle x, r \rangle) = \sum_x \mathcal{S}(x) e(-\langle \mathbf{g}(x), r \rangle) = p^{-1} \sum_{k \in \mathbb{F}_p} \sum_x e(k(\|x\| - j) - \langle \mathbf{g}(x), r \rangle) \\ &= p^{-1} \sum_{k \neq 0} e(-kj) \sum_x e(k\|x\| - \langle \mathbf{g}(x), r \rangle) = p^{-1} \sum_{k \neq 0} e(-kj) \sum_x e(k\|x\| - ax_1 - bx_2), \end{aligned}$$

where  $a, b \in \mathbb{F}_p$  are some constants depending of  $r$ . Completing the square and using the well-known formula

$$G(\alpha) := \sum_z e(\alpha z^2) = \left(\frac{\alpha}{p}\right) G(1),$$

where  $G(1) = \sum_z e(z^2)$  is the Gauss sum and  $\left(\frac{\alpha}{p}\right)$  is the Legendre symbol, we obtain

$$\widehat{\mathbf{g}(\mathcal{S})}(r) = \frac{G^2(1)}{p} \sum_{k \neq 0} e(-kj - ck^{-1}),$$

where  $c$  is some constant. Now applying  $|G(1)| = \sqrt{p}$  and the estimate for the Kloosterman sums [11], we get

$$\left| \sum_{k \neq 0} e(-kj - ck^{-1}) \right| \leq 2\sqrt{p}.$$

This completes the proof.  $\square$

For any set  $A \subseteq \Pi$  denote by  $f_A(x)$  the *balanced function* of the set  $A$ , that is  $f_A(x) = A(x) - |A|/|\Pi|$ . Clearly,  $\sum_x f_A(x) = 0$ . By  $I : \Pi \rightarrow \Pi$  denote the identity map.

**Theorem 3** *Let  $p$  be a sufficiently large prime number. Suppose that  $\mathbf{g}$  is an invertible affine transformation of  $\Pi$  such that  $\mathbf{g} - I$  is also invertible. Then for any two-coloring of the plane  $\Pi$  and any  $a \neq 0$  there is a monochromatic triple  $\{x, y, z\}$  such that  $y = x + s$ ,  $s \in \mathcal{S}_a$  and  $z = x + \mathbf{g}(s)$ .*

**Proof.** Let  $\mathcal{S} = \mathcal{S}_a$  and  $A, B$  be the colors of our coloring. We are interested into the quantity

$$\sigma(A) := \sum_x \sum_{s \in \mathcal{S}} A(x) A(x + s) A(x + \mathbf{g}(s)),$$

and similar for the color  $B$ . Let us rewrite the quantity  $\sigma(A)$  in terms of the balanced function of  $A$ . Put  $\delta_A = |A|/|\Pi|$ ,  $\delta_B = |B|/|\Pi|$ . Then because of the balanced function has zero mean, we get

$$\sigma(A) := \sum_x \sum_{s \in \mathcal{S}} (\delta_A + f_A)(x) (\delta_A + f_A)(x + s) (\delta_A + f_A)(x + \mathbf{g}(s)) =$$

$$\begin{aligned}
&= \delta_A^3 |\mathcal{S}| p^2 + \delta_A \left( \sum_x \sum_{s \in \mathcal{S}} f_A(x) f_A(x+s) + \sum_x \sum_{s \in \mathcal{S}} f_A(x) f_A(x+\mathbf{g}(s)) + \right. \\
&+ \left. \sum_x \sum_{s \in \mathcal{S}} f_A(x+\mathbf{g}(s)) f_A(x+s) \right) + \sum_x \sum_{s \in \mathcal{S}} f_A(x) f_A(x+s) f_A(x+\mathbf{g}(s)) = \\
&= \delta_A^3 |\mathcal{S}| p^2 + \delta_A (\sigma_1 + \sigma'_1 + \sigma''_1) + \sigma_2.
\end{aligned} \tag{7}$$

Let us estimate  $\sigma_1$ . By formulas (2), (3), we obtain

$$\sigma_1 = \sum_s \mathcal{S}(s) (f_A \circ f_A)(s) = p^{-2} \sum_r \widehat{\mathcal{S}}(r) |\widehat{f}_A(r)|^2$$

and thus, using the Parseval identity once more time as well as Lemma 2, we get

$$\sigma_1 = p^{-2} \sum_{r \neq 0} \widehat{\mathcal{S}}(r) |\widehat{f}_A(r)|^2 \leq 2\sqrt{p} \cdot p^{-2} \sum_r |\widehat{f}_A(r)|^2 \leq 2\sqrt{p} |A|.$$

So, it is negligible comparing the main term in (7). Now by the invertibility of  $\mathbf{g}$ , we have

$$\sigma'_1 = \sum_s \mathcal{S}(s) (f_A \circ f_A)(\mathbf{g}(s)) = \sum_s \mathcal{S}(\mathbf{g}^{-1}(s)) (f_A \circ f_A)(s)$$

and we can apply the arguments above because of one can use bound (6) of Lemma 2 instead of (5). Finally

$$\sigma''_1 = \sum_s \mathcal{S}(s) (f_A \circ f_A)(\mathbf{g}(s) - s) = \sum_s \mathcal{S}((\mathbf{g} - I)^{-1}(s)) (f_A \circ f_A)(s)$$

and by the invertibility of  $\mathbf{g} - I$  and in view of Lemma 2 we can estimate  $\sigma''_1$  similarly as  $\sigma'_1$ .

It remains to calculate the quantity  $\sigma_2$ . Using the inverse formula (1) it is easy to see that

$$\sigma_2 = \sigma_2(A) = p^{-4} \sum_{u,v} \widehat{f}_A(-u-v) \widehat{f}_A(u) \widehat{f}_A(v) \cdot \left( \sum_s \mathcal{S}(s) e(\langle s, u \rangle + \langle \mathbf{g}(s), v \rangle) \right).$$

Because of  $A(x) + B(x) = 1$  we have  $\widehat{f}_A(r) = -\widehat{f}_B(r)$  for all  $r \in \Pi$ . It follows that  $\sigma_2(A) + \sigma_2(B) = 0$ . Another way to see the fact is to check the identity  $f_A(x) + f_B(x) = 0$ . Whence, using Lemma 2 again, we obtain

$$\sigma(A) + \sigma(B) \geq |\mathcal{S}| p^2 (\delta_A^3 + \delta_B^3) - 6\sqrt{p}|A| - 6\sqrt{p}|B| \geq \frac{|\mathcal{S}| p^2}{4} - 6p^2 \sqrt{p} \geq \frac{p^3}{4} - 6.5p^2 \sqrt{p} > 0,$$

provided by  $p > 1000$ , say. This completes the proof.  $\square$

Because of two distinct points of  $\Pi$  can be transformed to another pair by a composition of an orthogonal transformation and a dilation (see e.g. [1], [5]) then we obtain two immediate consequences of the theorem above.

**Corollary 4** *Let  $p$  be a sufficiently large prime number,  $p \equiv -1 \pmod{4}$ , and three points  $A, B, C \in \Pi$  form a non-equilateral triangle. Then for any two-coloring of the plane  $\Pi$  there is a monochromatic triangle congruent to  $\triangle ABC$ .*

**Proof.** First of all note that any triangle has a pair of sides such that the quotient of its "lengths" is a quadratic residue. Let  $\|A - B\| = a$ ,  $\|A - C\| = b$ , and  $a/b$  be a quadratic residue. Then there is an affine transformation  $\mathbf{g}$  (which is a composition of an orthogonal map of  $\Pi$  and a dilation, see [5]) such that  $\mathbf{g}(A) = A$ ,  $\mathbf{g}(B) = C$ . It is easy to see that both maps  $\mathbf{g}$  and  $\mathbf{g} - I$  are invertible. Indeed, if  $\mathbf{g}$  is not invertible then there is  $x \neq 0$  such that  $\|x\| = 0$ . But in view of the assumption  $p \equiv -1 \pmod{4}$ , we derive  $x = 0$  with a contradiction (note that in the case of  $p \equiv 1 \pmod{4}$  there is  $i \in \mathbb{F}_p$  such that  $i^2 \equiv -1 \pmod{p}$  and hence there are  $x \neq 0$  with  $\|x\| = 0$ ). Finally, if  $\mathbf{g} - I$  is not invertible then  $\triangle ABC$  is equilateral and for some  $x$  one has  $\mathbf{g}x = x$  and hence  $\mathbf{g}$  is a mirror symmetry (in the case there are some restrictions on the length of the side  $a$  of  $\triangle ABC$  as  $a$  is nonresidual and  $a + 1$  is residual but we miss them). This completes the proof.  $\square$

**Corollary 5** *Let  $p$  be a sufficiently large prime number. Then for any two-coloring of the plane  $\Pi$  and any  $a, b \neq 0$  such that  $a/b$  is a quadratic residue there is a monochromatic collinear triple  $\{x, y, z\}$  with  $\|y - x\| = a$ ,  $\|z - y\| = b$ .*

**Problem.** Is it possible to find larger monochromatic configurations from  $\Pi$  in the spirit of papers [2], [12]?

### 3 Euclidian plane

In the section we consider the case of the usual euclidian plane and try to obtain an analog of Theorem 3. The proof follows the arguments from [9] as well as the approach (and the notation) from the previous section.

Put  $\Pi = \mathbf{R}^2$  and let  $A \subseteq \Pi$  be a measurable set. By the *upper density* of  $A$  define

$$\bar{\delta}_A := \limsup_{T \rightarrow +\infty} \frac{\text{Vol}(A \cap [-T, T]^2)}{(2T)^2}. \quad (8)$$

A measurable, complex valued function  $f : \Pi \rightarrow \mathbb{C}$  is called *periodic* if there is a basis  $b_1, b_2 \in \Pi$  such that for all  $\alpha_1, \alpha_2 \in \mathbb{Z}$  one has  $f(x + \alpha_1 b_1 + \alpha_2 b_2) = f(x)$ . The set  $L = \{\alpha_1 b_1 + \alpha_2 b_2 : \alpha_1, \alpha_2 \in \mathbb{Z}\}$  is called the *period lattice* of  $f$  and  $L^* = \{u \in \Pi : \langle u, x \rangle \in \mathbb{Z}, \forall x \in L\}$  is called the *dual lattice* of  $L$ . Here  $\langle \cdot, \cdot \rangle$  is the usual scalar product onto  $\Pi$ . If the characteristic function of a measurable set  $A$  is periodic then it is easy to see that  $\limsup$  in (8) can be replaced by a simple limit.

We have a scalar product on the space of periodic functions

$$\langle f, g \rangle = \lim_{T \rightarrow +\infty} \frac{1}{(2T)^2} \int_{[-T, T]^2} f(x) \overline{g(x)} dx.$$

The Fourier transform of a (periodic) function  $f$  is given by the formula  $\widehat{f}(u) = \langle f(x), e^{i\langle u, x \rangle} \rangle$ . It is easy to check that the support of Fourier transform of a periodic function  $f$  belongs to  $2\pi L^*$ . In particular, the support is a discrete set.

The Bessel function of the first kind  $J_\nu(z)$  is the series (see e.g. [8])

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}. \quad (9)$$

It is well-known that

$$J_0(\|u\|) = \frac{1}{2\pi} \langle \mathcal{S}_1(x), e^{iux} \rangle, \quad (10)$$

where  $J_0$  is the zeroth Bessel function and

$$\mathcal{S}_a = \{x \in \Pi : \|x\| := \sqrt{x_1^2 + x_2^2} = a\}$$

is a circle of radius  $a$ .

**Theorem 6** *Let  $a > 0$  and  $\kappa > 0$  be real numbers. Suppose that for all  $t \geq 0$  one has*

$$J_0(t) + J_0(\kappa t) + J_0((1 + \kappa)t) > -1. \quad (11)$$

*Then for any measurable coloring of the plane  $\Pi$  into two colors there is a monochromatic collinear triple  $\{x, y, z\}$  such that  $y \in [x, z]$  and  $\|y - x\| = a$ ,  $\|z - y\| = \kappa a$ .*

**Proof.** We follow the arguments of the proof of Theorem 3. Let  $\mathcal{S} = \mathcal{S}_a$  be the circle of radius  $a$  and  $A, B$  be the colors of upper densities  $\bar{\delta}_A, \bar{\delta}_B$ . We suppose that  $A$  and  $B$  do not contain collinear triples  $\{x, y, z\}$  such that  $y \in [x, z]$  and  $\|y - x\| = a$ ,  $\|z - y\| = \kappa a$ .

One can assume that  $A(x)$  and  $B(x)$  are periodic functions. Indeed, choose  $T$  is large enough that  $[-T + (1 + \kappa)a, T - (1 + \kappa)a]^2 / (2T)^2$  is sufficiently close to 1 and such that  $\text{Vol}(A \cap [-T, T]) / (2T)^2$  is sufficiently close to  $\bar{\delta}_A$ . After that construct a periodic tiling of  $\mathbf{R}^2$  with copies of  $A \cap [-T + (1 + \kappa)a, T - (1 + \kappa)a]^2$  and  $B \cap [-T + (1 + \kappa)a, T - (1 + \kappa)a]^2$ , translating the copies by the points of lattice  $2T\mathbb{Z}$ . Denote the obtained new colors as  $A_*$  and  $B_*$ . Clearly,  $\delta_{A_*}$  can be chosen is close to  $\bar{\delta}_A$  and that  $\delta_{A_*} + \delta_{B_*} = 1$ . Note also that  $A_*, B_*$  do not contain collinear triples  $\{x, y, z\}$  such that  $y \in [x, z]$  and  $\|y - x\| = a$ ,  $\|z - y\| = \kappa a$ .

As in the proof of Theorem 3 consider the quantity  $\sigma(h_1, h_2, h_3)$ , which is trilinear by three arguments  $h_1, h_2, h_3$ , namely,

$$\begin{aligned} \sigma(A_*, A_*, A_*) &= \sigma(A_*) := \\ \lim_{T \rightarrow +\infty} \frac{1}{(2T)^2} \int \int_{s \in \mathcal{S}} (\delta_{A_*} + f_{A_*})(x) (\delta_{A_*} + f_{A_*})(x + s) (\delta_{A_*} + f_{A_*})(x - \kappa s) dx ds, \end{aligned}$$

where again  $f_{A_*}(x) = A_*(x) - \delta_{A_*}$  is the balanced function of  $A_*$ . We have

$$\sigma(A_*) := 2\pi a \delta_{A_*}^3 + \delta_A (\sigma(f_{A_*}, f_{A_*}, 1) + \sigma(f_{A_*}, 1, f_{A_*}) + \sigma(1, f_{A_*}, f_{A_*})) + \sigma(f_{A_*}). \quad (12)$$

As in the proof of Theorem 3 the following holds  $\sigma(f_{A_*}) + \sigma(f_{B_*}) = 0$  and thus we need to bound the remain three quantities in (12). Clearly,

$$\sigma(f_{A_*}, f_{A_*}, 1) = \langle f_{A_*} \circ f_{A_*}, \mathcal{S} \rangle.$$

Using the Fourier transform, we get

$$\sigma(f_{A_*}, f_{A_*}, 1) = \sum_{u \in \mathbf{R}^2} |\widehat{f_{A_*}}(u)|^2 \widehat{\mathcal{S}}(u). \quad (13)$$

As we noted before the sum in (13) is actually taking over a discrete set. Putting

$$\alpha(t) := \sum_{u \in \mathbf{R}^2 : \|u\|=t} |\widehat{f_{A_*}}(u)|^2 \geq 0,$$

we obtain by (10)

$$\sigma(f_{A_*}, f_{A_*}, 1) = 2\pi a \sum_{t \geq 0} J_0(at) \alpha(t).$$

Here we have used the formula

$$\widehat{\mathcal{S}}_b(u) = b \widehat{\mathcal{S}}_1(bu) = 2\pi b J_0(\|bu\|),$$

where  $b > 0$  is an arbitrary. Similarly,

$$\sigma(f_{A_*}, 1, f_{A_*}) = \langle (f_{A_*} \circ f_{A_*})(\kappa s), \mathcal{S}(s) \rangle = 2\pi a \sum_{t \geq 0} J_0(\kappa at) \alpha(t),$$

and

$$\sigma(1, f_{A_*}, f_{A_*}) = \langle (f_{A_*} \circ f_{A_*})((1 + \kappa)s), \mathcal{S}(s) \rangle = 2\pi a \sum_{t \geq 0} J_0((1 + \kappa)at) \alpha(t).$$

Thus

$$\sigma(f_{A_*}, f_{A_*}, 1) + \sigma(f_{A_*}, 1, f_{A_*}) + \sigma(1, f_{A_*}, f_{A_*}) \geq 2\pi a \sum_{t \geq 0} \alpha(t) (J_0(at) + J_0(\kappa at) + J_0((1 + \kappa)at)).$$

By  $J$  define the quantity  $J = \min_{t \geq 0} (J_0(at) + J_0(\kappa at) + J_0((1 + \kappa)at))$ . Applying the Parseval identity and the observation  $\widehat{A_*}(0) = \delta_{A_*}$ , we have

$$\sum_{t \geq 0} \alpha(t) = \sum_u |\widehat{A_*}(u)|^2 - \delta_{A_*}^2 = \delta_{A_*} - \delta_{A_*}^2. \quad (14)$$

Returning to (12) and combining it with the last formula, we obtain

$$(2\pi a)^{-1} (\sigma(A_*) + \sigma(B_*)) \geq \delta_{A_*}^3 + \delta_{B_*}^3 + J(\delta_{A_*}^2 - \delta_{A_*}^3) + J(\delta_{B_*}^2 - \delta_{B_*}^3) = (\delta_{A_*}^3 + \delta_{B_*}^3)(1 - J) + (\delta_{A_*}^2 + \delta_{B_*}^2)J.$$

Because of  $\delta_{A_*} + \delta_{B_*} = 1$  the optimization gives us

$$(2\pi a)^{-1} (\sigma(A_*) + \sigma(B_*)) \geq \frac{J + 1}{4} > 0.$$

Here we have used condition (11). This completes the proof.  $\square$

**Corollary 7** *Let  $a > 0$  be a real number. Then for any measurable coloring of the plane  $\Pi$  into two colors there is a monochromatic collinear triple  $\{x, y, z\}$  such that  $y \in [x, z]$  and  $\|y - x\| = \|z - y\| = a$ .*

**Proof.** By Theorem 6, we need to estimate  $\min_{t \geq 0} (2J_0(at) + J_0(2at)) = \min_{t \geq 0} (2J_0(t) + J_0(2t))$ . Using Maple, say, one can calculate  $\min_{t \in [0, 50]} (2J_0(t) + J_0(2t)) \geq -0.74$ . For  $t > 50$ , applying a crude upper bound  $|J_\nu(t)| \leq |t|^{-1/3}$ ,  $\nu \geq 0$  (see e.g. [7]), we insure that the minimum is strictly greater than  $-1$  for all  $t \geq 0$ . This concludes the proof.  $\square$

Below we will deal with affine transformations  $\mathbf{g}$  of the form  $\mathbf{g} = D_\omega \circ R$ , where  $R$  be a rotation and  $D_\omega$  be a dilation by some  $\omega > 0$ . Let us note a simple lemma about such  $\mathbf{g}$ .

**Lemma 8** *Let  $\mathbf{g} = D_\omega \circ R$ , where  $D_\omega$  be a dilation by  $\omega$  and  $R$  be a rotation by  $\varphi$ . Then  $\mathbf{g} - I$  has the same form  $D_{\omega'} \circ R'$ , where  $\omega' = \sqrt{\omega^2 - 2\omega \cos \varphi + 1}$  and  $R'$  is another rotation.*

**Proof.** To obtain the result we need to solve the system of equations  $\omega \cos \varphi - 1 = \omega' \cos \varphi'$ ,  $\omega' \sin \varphi' = \omega \sin \varphi$  in variables  $\omega', \varphi'$ . Taking a square and a summation give us

$$(\omega')^2 = \omega^2 - 2\omega \cos \varphi + 1 \geq 0$$

and thus  $\sin \varphi' = \omega/\omega' \cdot \sin \varphi$ ,  $\cos \varphi' = (\omega \cos \varphi - 1)/\omega'$ . One can check that the modules of  $\omega/\omega' \cdot \sin \varphi$  as well as  $(\omega \cos \varphi - 1)/\omega'$  do not exceed 1 and hence  $\varphi'$  exists. This completes the proof.  $\square$

Similarly to Theorem 6 as well as Theorem 3, one can obtain the following general result, which is however not so wide as Theorem 3.

**Theorem 9** *Let  $a > 0$  and  $\omega > 0$  be real numbers. Let also  $\mathbf{g} = D_\omega \circ R$  be an affine transformation of  $\Pi$ , where  $R$  be a rotation and  $D_\omega$  be a dilation by  $\omega$ . Suppose that for all  $t \geq 0$  one has*

$$J_0(t) + J_0(\omega t) + J_0 > -1, \quad (15)$$

where  $J_0 = \min_{t \geq 0} J_0(t) = -0.4027593957\dots$ . Then for any measurable coloring of the plane  $\Pi$  into two colors there is a monochromatic collinear triple  $\{x, y, z\}$  such that  $y = x + s$ ,  $s \in \mathcal{S}_a$  and  $z = x + \mathbf{g}(s)$ . More precisely, if  $R$  is a rotation by  $\varphi$  then condition (15) can be replaced by

$$J_0(t) + J_0(t\omega) + J_0(t\sqrt{\omega^2 - 2\omega \cos \varphi + 1}) > -1. \quad (16)$$

**Proof.** We use the notation and the arguments of the proof of Theorem 6. Then

$$\begin{aligned} \sigma(A_*, A_*, A_*) &= \sigma(A_*) := \\ \lim_{T \rightarrow +\infty} \frac{1}{(2T)^2} \int \int_{s \in \mathcal{S}} (\delta_{A_*} + f_{A_*})(x) (\delta_{A_*} + f_{A_*})(x + s) (\delta_{A_*} + f_{A_*})(x + \mathbf{g}(s)) dx ds, \\ &= 2\pi a \delta_{A_*}^3 + \delta_A(\sigma(f_{A_*}, f_{A_*}, 1) + \sigma(f_{A_*}, 1, f_{A_*}) + \sigma(1, f_{A_*}, f_{A_*})) + \sigma(f_{A_*}). \end{aligned}$$

Again, we need to estimate  $\sigma(f_{A_*}, f_{A_*}, 1)$ ,  $\sigma(f_{A_*}, 1, f_{A_*})$ ,  $\sigma(1, f_{A_*}, f_{A_*})$ . The first quantity is the same as in the proof of Theorem 6. The second one equals

$$\sigma(f_{A_*}, 1, f_{A_*}) = \langle (f_{A_*} \circ f_{A_*})(\mathbf{g}(s)), \mathcal{S}(s) \rangle = \langle (f_{A_*} \circ f_{A_*})(s), \mathcal{S}(\mathbf{g}^{-1}(s)) \rangle \cdot \det(\mathbf{g})^{-1}. \quad (17)$$



As we know for any  $b > 0$  one has

$$\widehat{\mathcal{S}}_b(u) = b\widehat{\mathcal{S}}_1(bu) = 2\pi b J_0(\|bu\|). \quad (18)$$

Hence

$$\langle \mathcal{S}(\mathbf{g}^{-1}(s)), e^{i\langle u, s \rangle} \rangle = 2\pi a J_0(\omega a \|u\|) \cdot \det(\mathbf{g}). \quad (19)$$

In terms of quantities  $\alpha(t)$  it follows that

$$\sigma(f_{A_*}, 1, f_{A_*}) = 2\pi a \sum_{t \geq 0} \alpha(t) J_0(\omega a t).$$

Finally, in the estimation of the third term  $\sigma(1, f_{A_*}, f_{A_*})$  the quantity  $(\mathbf{g} - I)^{-1}$  appears. Hence (see the proof of Theorem 3), we get

$$\sigma(f_{A_*}, 1, f_{A_*}) = \sum_{u \in \mathbf{R}^2} |\widehat{f_{A_*}}(u)|^2 \langle \mathcal{S}((\mathbf{g} - I)^{-1}(s)), e^{i\langle u, s \rangle} \rangle. \quad (20)$$

Unfortunately, if  $u$  runs over a circle then  $(\mathbf{g} - I)^{-1}(u)$  do not belong to a circle in the case of general transformation  $\mathbf{g}$  (but it is so in the case of Theorem 6 when  $\{x, y, z\}$  are collinear). Nevertheless we estimate (20) with help of (14) crudely as

$$\sigma(f_{A_*}, 1, f_{A_*}) \geq 2\pi a J_0 \cdot (\delta_{A_*} - \delta_{A_*}^2).$$

Combining all bounds, we obtain

$$(2\pi a)^{-1}(\sigma(A_*) + \sigma(B_*)) \geq \frac{J + J_0 + 1}{4} > 0,$$

where  $J = \min_{t \geq 0} (J_0(at) + J_0(\omega a t)) = \min_{t \geq 0} (J_0(t) + J_0(\omega t))$ . Thus, we have proved (15) and it remains to obtain (16). In the case apply Lemma 8, combining with formula (20) and calculations in (17)—(19). This completes the proof.  $\square$

**Remark 10** *It is well-known that there is a measurable two-coloring of the plane having no monochromatic equilateral triangle of an arbitrary side  $a > 0$ , see [3]. If we try to apply Theorem 9 in the case then a Maple calculation gives us*

$$\min_{t \geq 0} (2J_0(t)) + J_0 = 3J_0 = -1.208278187...$$

*It is rather close to the required  $-1$ .*

There is a series of results, see e.g. [10] where the existence of monochromatic triangle with some restrictions on the lengths of the sides and the angles was obtained for any (non-necessary measurable) coloring. For example, in [3] the authors proved that any monochromatic triangle with the smallest side 1 and the angles in the ratio  $1 : 2 : 3$ , more generally, in the ratio  $n : (n + 1) : (2n + 1)$ ,  $1 : 2n : (2n + 1)$  and so on can be found. Concluding the section we note that in our Theorem 9 one does not need to know any angles but just the ratio of the lengths of an arbitrary two sides of the triangle. For example, one can show that for  $\omega = 2$  the minimum in (15) is greater than  $-0.86$  and hence any monochromatic triangle with the ratio of the sides  $1 : 2$  appears.

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I.D. Shkredov  
 Steklov Mathematical Institute,  
 ul. Gubkina, 8, Moscow, Russia, 119991  
 and  
 IITP RAS,  
 Bolshoy Karetny per. 19, Moscow, Russia, 127994  
 ilya.shkredov@gmail.com